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Emerging Holography

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Abstract

We rederive AdS/CFT predictions for infrared two-point functions by an entirely four dimensional approach, without reference to holography. This approach, originally due to Migdal in the context of QCD, utilizes an extrapolation from the ultraviolet to the infrared using a Padé approximation of the two-point function. We show that the Padé approximation and AdS/CFT give the same leading order predictions, and we discuss including power corrections such as those due to condensates of gluons and quarks in QCD. At finite order the Padé approximation provides a gauge invariant regularization of a higher dimensional gauge theory in the spirit of deconstructed extra dimensions. The radial direction of anti-de Sitter space emerges naturally in this approach.

1 Introduction

The AdS/CFT correspondence provides an intriguing relationship between a weakly coupled $d+1$ dimensional gravity theory and a strongly coupled d dimensional conformal field theory (CFT) [1, 2, 3]. The evidence for a holographic relationship between five-dimensional (5D) gravity theories and four-dimensional (4D) CFTs is by now relatively well established, and articulated, for example, in [4]. This led to several applications, particularly with regard to interpreting versions of the Randall-Sundrum model [5, 6] as an approximate 4D CFT whose conformal invariance is broken in the infrared (IR). The IR breaking of conformal invariance implies a discrete spectrum of CFT resonances (on the 5D side: spectrum of Kaluza-Klein excitations), and thus we have a 4D theory that really has particles and an S-matrix, allowing us to exploit this relationship for phenomenology.

Recent phenomenological applications of the AdS/CFT correspondence are the proposals of a simple holographic dual to low energy QCD [7, 8, 9, 10]. These models have allowed for the calculation of properties of mesons and baryons, given relatively few input parameters, with results that so far are remarkably consistent with experimental data to within 15%. Physics such as chiral symmetry breaking and confinement that are associated with the existence of an IR brane is put in by hand, following the AdS/CFT dictionary. The discrete spectrum of Kaluza-Klein excitations of bulk fields become the composites of QCD. These models are referred to as AdS/QCD models; several extensions and simplifications of the AdS/QCD models have been proposed in [11, 12, 13, 14, 15]. For some of the earlier attempts on applying AdS/CFT to QCD, see [16, 17, 18, 19]. What is most surprising about the success of AdS/QCD is the success itself. The AdS/CFT correspondence is a duality between a CFT and a string theory in a particular spacetime background. If N_c and g^2N_c are large, where N_c and g are the number of colors and gauge coupling constant respectively, then stringy physics decouples and a classical field theory coupled to gravity remains. However, in QCD neither N_c nor g^2N_c is large, which suggests that even if there is a holographic dual to QCD, the full string theory may be required to make useful predictions. Nevertheless, it turns out that a simple model using a 5D field theory in AdS space works surprisingly well.

One of our motivations in this paper is to address why the AdS/CFT approach has worked so well for QCD, by comparing with an orthogonal approach that yields similar results. Shifman [20] and Voloshin [21] recently pointed out that in the '70s Migdal computed masses of mesons in large N_c QCD using a Padé approximation to QCD current-current correlators, and found them to be roots of Bessel functions [22, 23]. (See [24] for spectrum of QCD in 2+1 dimensions involving roots of Bessel functions.) The technique amounts to approximating the current-current correlator in the deep Euclidean regime by a ratio of two polynomials of degree N (not to be confused with N_c the number of colors). For large Euclidean momenta, correlators are dominated by short-distance physics and exhibit conformal behavior. The Padé approximant is then analytically continued to the low energy regime where resonance physics dominates. Physically, the approach is tantamount to finding the best fit to high energy data with a finite set of resonances. At low energies and large N, Migdal found that the Padé polynomials could be approximated by Bessel functions, giving the masses of the mesons as roots of Bessel functions.

The appearance of Bessel functions in this approach is tantalizingly similar to AdS/CFT.

In this paper we will make explicit the relation between the Padé approximation of current-current correlators and the AdS/CFT correspondence, and show that both approaches give identical leading order predictions. The comparison requires a careful treatment of the Padé approximation in which a UV regulator scale μ and the polynomial degree N are sent to infinity while the ratio N/μ is held fixed. The length scale N/μ is identified with the location of the IR brane in the AdS/CFT approach, which is also related to the confinement scale. We emphasize that just like in AdS/QCD, the IR scale is put in by hand, and is *a priori* independent of the chiral symmetry breaking condensates. Nevertheless, the Padé approximation is remarkably powerful in being able to manifest other aspects of AdS/CFT. For instance, a very recent paper [15] incorporated power corrections arising from condensates of quarks and gluons [25, 26] in the AdS/QCD framework. We show that the Padé approximation can also accommodate power corrections, and the result continues to agree with the AdS/CFT approach. Finally, we show how the radial direction of 5D anti-de Sitter space emerges as the order of the Padé approximation increases, by analogy with deconstructed extra dimensions [27, 28].

The outline of this paper is as follows. In Sec. 2 we review the computation of conformal two-point functions in AdS/CFT. In Sec. 3, we describe Migdal's Padé approximation for the current-current correlation function and show that in the low-energy, large N limit it gives the same expression as that obtained from AdS/CFT. In Sec. 4 we consider the breaking of the conformal symmetry through power corrections, and again show that the two approaches agree. In Sec. 5 we show how the Padé approximation gives rise to a gauge invariant regularization of warped 5D gauge theories by analogy with deconstructed extra dimensions. Finally, we conclude in Sec. 6.

2 The AdS/CFT Correspondence

The prototypical example of the AdS/CFT correspondence is the duality between 4D $\mathcal{N}$=4 $SU(N_c)$ super Yang-Mills theory and Type IIB string theory compactified on $\mathrm{AdS}_5\times\mathrm{S}_5$. In a limit which involves taking the rank of the gauge group and the 't Hooft coupling large, string theory effects decouple and classical Type IIB supergravity on $\mathrm{AdS}_5\times\mathrm{S}_5$ remains. With this simplification it becomes possible to analytically calculate correlation functions in the 4D gauge theory from supergravity. The basic dictionary relating correlation functions in the CFT to supergravity on an AdS background was found in [2, 3], and many tests have since been performed. In this section we briefly review the computation of the CFT two-point function for vector currents in AdS/CFT. Each operator in the CFT corresponds to a field in anti-de Sitter space of one dimension higher, whose boundary value is set equal to the source for the corresponding operator. More precisely, if we consider the AdS metric in the form,

$$ds^2 = \frac{1}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu - dz^2\right), \tag{1}$$

where $\eta_{\mu\nu} = \mathrm{Diag}(1,-1,-1,-1)$, the prescription in [2, 3] states that

$$\langle e^{i\int d^4x\,\phi_0(x)\mathcal{O}(x)}\rangle_{\mathrm{CFT}} = e^{iS_{\mathrm{AdS}}[\phi^{\mathrm{cl}}]}\Big|_{\phi^{\mathrm{cl}}(x,z=0)=\phi_0}, \tag{2}$$

where $S_{\mathrm{AdS}}[\phi^{\mathrm{cl}}]$ is the classical action in the AdS space and ϕ^{cl} is a solution to the equation of motion whose boundary value is fixed to be the source $\phi_0(x)$ (up to a conformal rescaling). In practice one introduces an UV regulator $z = \epsilon$ and then takes $\epsilon \to 0$. For a vector current such as $J^\mu = \bar{q}\gamma^\mu q$ in the CFT, there is a corresponding bulk gauge field $A_M(x,z)$ whose boundary value is the source for J^μ. The 5D action in AdS space is

$$S_{\mathrm{AdS}} = -\int d^4x dz \sqrt{-g}\;\frac{1}{4g_5^2}F_{MN}F^{MN}, \tag{3}$$

where the capital roman letters $M, N = 0, 1, 2, 3, z$. We have neglected a bulk mass term for A_M since we are considering conserved currents. According to the AdS/CFT correspondence, to calculate the current-current correlator, we calculate the 5D action on a solution to the equations of motion for the corresponding gauge field such that the 5D gauge field at the UV boundary has the profile of the 4D source of the current. We will consider a finite AdS space with an infrared boundary at $z = z_0$. The profile of the 5D gauge field satisfying those boundary conditions is the bulk-to-boundary propagator, which we call $V(q,z)$. Varying the action (twice) with respect to the boundary source gives the current-current correlator. We impose the $A_z = 0$ gauge and Fourier-transform the gauge field in four dimensions:

$$A_\mu(q,z) = \frac{1}{V(q,\epsilon)}\widetilde{A}_\mu(q)V(q,z), \tag{4}$$

where $\widetilde{A}_\mu(q)$ is the Fourier-transformed current source. The boundary condition $A_\mu(q,\epsilon) = \widetilde{A}_\mu(q)$ is built into (4). The boundary condition at $z = z_0$ is not completely predetermined, but for definiteness we assume Neumann boundary conditions there, $\partial_z V(q,z_0) = 0$, corresponding to the gauge invariant condition $F_{\mu z}(x,z_0) = 0$. The equations of motion for the transverse part of the gauge field are,

$$z\,\partial_z\left(\frac{1}{z}\partial_z V(q,z)\right) + q^2\,V(q,z) = 0, \tag{5}$$

which, given the boundary conditions, leads to

$$V(q,z) = qz\left(Y_0(qz_0)J_1(qz) - J_0(qz_0)Y_1(qz)\right). \tag{6}$$

Evaluating the action on the solution leaves only the the boundary term at the UV

$$S_{\mathrm{AdS}} = -\frac{1}{2g_5^2}\int d^4q\,\widetilde{A}_\mu(q)\widetilde{A}^\mu(-q)\left(\frac{1}{z}\frac{\partial_z V(q,z)}{V(q,z)}\right)_{z=\epsilon}. \tag{7}$$

If we write the Fourier-transformed vector current two-point function as,

$$\int d^4x\,e^{iq\cdot x}\,\langle J_\mu(x)J_\nu(0)\rangle = \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\Sigma(q^2), \tag{8}$$

then functionally differentiating the action (7) with respect to the source $\widetilde{A}_\mu(q)$ yields the vector current-current correlator determined by AdS/CFT:

$$\begin{aligned}
\Sigma(q^2) &= -\frac{1}{g_5^2}\,\frac{1}{z}\,\frac{\partial_z V(q,z)}{V(q,z)}\bigg|_{z=\epsilon\to 0} \\
&= -\frac{1}{g_5^2}\,\frac{1}{z}\,q\,\frac{Y_0(qz_0)J_0(qz)-J_0(qz_0)Y_0(qz)}{Y_0(qz_0)J_1(qz)-J_0(qz_0)Y_1(qz)}\bigg|_{z=\epsilon} \\
&\to \frac{q^2}{g_5^2}\,\frac{\log(q\epsilon)J_0(qz_0)-(\pi/2)Y_0(qz_0)}{J_0(qz_0)},
\end{aligned} \tag{9}$$

where we have retained the leading non-vanishing contribution from each of the Bessel functions in the limit $q\epsilon \to 0$:

$$J_0(q\epsilon)\to 1, \quad J_1(q\epsilon)\to \frac{q\epsilon}{2}\to 0, \quad Y_0(q\epsilon)\to \frac{2}{\pi}\log(q\epsilon), \quad Y_1(q\epsilon)\to -\frac{2}{\pi}\frac{1}{q\epsilon}. \tag{10}$$

Note that we have neglected terms in (9) that can be absorbed into the re-definition of ϵ, such as the Euler constant γ_E in the expansion of $Y_0(q\epsilon)$. From (9) we see that the masses of the resonances in the CFT, given by the simple poles in $\Sigma(q^2)$, are determined by roots of the Bessel function $J_0(qz_0)$. On the other hand, if we simultaneously take $qz_0 \gg 1$, then the spectrum becomes continuous and the vector polarization $\Sigma(q^2)$ turns into, up to contact terms,

$$\Sigma(q^2) \to \frac{1}{2g_5^2}\,q^2\log(q^2\epsilon^2). \tag{11}$$

Here we see explicitly that ϵ plays the role of an UV regulator. In the context of AdS/QCD, one matches (11) onto the quark bubble calculation in QCD at large Euclidean momentum, which gives,

$$g_5^2 = \frac{12\pi^2}{N_c} \tag{12}$$

in units of the AdS curvature [9, 10].

The expression in (12) serves as a good point to discuss why a strict reading of the AdS/CFT correspondence would suggest that the models in AdS/QCD shouldn't have worked so well. In QCD $N_c = 3$ which then suggests that the 5D gauge coupling is strong. Therefore higher order corrections in the AdS side may be as important as the tree-level results. Furthermore, on the QCD side the 't Hooft coupling $g_4^2 N_c \sim 1$, which would suggest stringy effects cannot be decoupled in the AdS space. Again, a simple model of a 5D gauge field does not appear to be justified *a priori*, yet somehow produces accurate results.

In order to include corrections due to higher dimension operators in the operator product expansion (OPE) of the two-point function (power corrections), we can add to the action higher dimension operators and additional bulk fields [9, 10]. For example, including the dilaton in the theory allows couplings which would mimic the inclusion of higher dimension operators involving the QCD field strength. Alternatively, a simpler modification of the theory which also captures the effects of power corrections in the two-point function is to allow deviations in the geometry towards the boundary [15]. These modifications directly encode breaking of conformal

symmetry away from the UV, which is also a consequence of chiral symmetry breaking. We will see in Sec. 4 that Migdal's Padé approximation approach to estimating current correlators can be easily generalized to include higher dimension operators in the OPE.

3 The Padé Approximation and Vector Mesons

In the '70s Migdal proposed a systematic procedure for calculating masses of mesons in large N_c QCD [22, 23], which approximates the two-point function in the deep Euclidean region by a rational function and then analytically continues into the infrared domain. Mathematically this procedure amounts to performing the Padé approximation. The rationale behind Migdal's approach relies on properties of large N_c, in particular the vanishing of instanton corrections to current correlators. It is unclear how nonperturbative corrections should be added in the $1/N_c$ expansion, so the justification for the Padé approximation is limited. Nonetheless, ignoring the effects of QCD condensates, Migdal found that the masses of the mesons in large N_c QCD are proportional to roots of certain Bessel functions, which is quite intriguing from the modern perspective, as was recently emphasized by Shifman [20] and Voloshin [21].

It is the purpose of this section to first give an overview of Migdal's approach from nearly three decades ago and make precise the relation between AdS/CFT and the Padé approximation of correlation functions. We will focus our discussion on the two-point function of conserved vector currents in a CFT, although the construction can be generalized to arbitrary conformal tensors [22, 23]. Two-point correlators of operators with arbitrary conformal dimension in the UV generally involve expressions of the form,

$$\Sigma_\nu(q^2) = (q^2)^\nu \log \frac{q^2}{\mu^2}. \tag{13}$$

The scale μ is a renormalization scale and $\nu = \Delta - 2$, where Δ is the conformal dimension of the operator involved. For a conserved vector current, $\Delta = 3$ and $\nu = 1$, which is consistent with (11). We will consider the dimensionless quantity

$$f_0(t) = \log t \tag{14}$$

so that $f_0(q^2/\mu^2) = (q^2)^{-\nu}\Sigma_\nu(q^2)$. Migdal proposed that one find a ratio of two polynomials of degree N respectively, which reproduces the first $2N+1$ terms in the Taylor expansion of $f_0(t)$ with respect to an arbitrary subtraction point, which we choose to be $t = -1$, *i.e.* $q^2 = -\mu^2$. Mathematically this procedure is known as the Padé approximation, which allows functions with possible poles and branch cuts to be well-represented by polynomials.

To be precise, we are looking for two polynomials $R_N(t)$ and $Q_N(t)$ of degree N that satisfy

$$f_0(t) - \frac{R_N(t)}{Q_N(t)} = \mathcal{O}\left((t+1)^{2N+1}\right), \tag{15}$$

which in turn implies

$$\frac{d^m}{dt^m}\left[Q_N f_0(t) - R_N\right]\Big|_{t=-1} = 0, \quad m = 0, 1, \cdots, 2N. \tag{16}$$

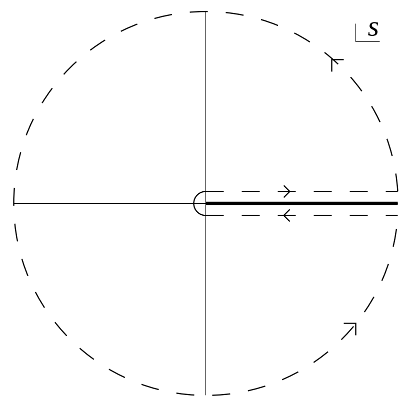

Figure 1: Contour used to evaluate derivatives of $Q_N f_0(t)$ in (18).

Notice that, since $R_N(t)$ is only of degree N, it is determined completely by the first $N+1$ equations in (16) for $m = 0, 1, \cdots, N$, if Q_N is known; the remaining N equations determine the denominator function $Q_N(t)$ up to an overall constant (corresponding to a simultaneous rescaling of R_N and Q_N):

$$\left.\frac{d^m}{dt^m} Q_N f_0(t)\right|_{t=-1} = 0, \quad m = N+1, \cdots, 2N. \tag{17}$$

To solve for Q_N we can analytically continue (17) into the complex plane. The function $f_0(t)$ has a branch cut which is taken to be along the positive real axis. This branch cut is approximated by a series of poles in the Padé approximation, and the resonance masses are interpreted as the locations of the poles. The imaginary part of the function $f_0(t)$ jumps by 2π across the branch cut. At any location in the complex t-plane away from the positive real axis, derivatives of the function $Q_N f_0(t)$ are given by the Cauchy integral formula along the path in Fig 1. The integral over the large and small circles vanish (for the orders of derivative we are considering), and the integrals above and below the branch cut cancel except for the change in the imaginary part of f_0. What remains is given by,

$$\left.\frac{d^m}{dt^m} Q_N f_0(t)\right|_{t=-1} = \frac{m!}{2\pi} \int_0^\infty ds \, \frac{Q_N \, \Delta \mathrm{Im} f_0(s)}{(s+1)^{m+1}} = 0, \quad m = N+1, \cdots, 2N, \tag{18}$$

where $\Delta \mathrm{Im} f_0(t) = \mathrm{Im} f_0(t + i\epsilon) - \mathrm{Im} f_0(t - i\epsilon)$ is the change in $f_0(t)$ across the branch cut. On the other hand, from (16) we have

$$\begin{aligned} R_N &= Q_N f_0(t) - \sum_{n=N+1}^{\infty} \frac{1}{n!} \left.\frac{d^n}{dt^n} Q_N f_0(t)\right|_{t=-1} (t+1)^n \\ &= Q_N f_0(t) - \sum_{n=N+1}^{\infty} \int_0^\infty ds \, \frac{1}{2\pi} \frac{Q_N \, \Delta \mathrm{Im} f_0(s)}{(s+1)^{n+1}} (t+1)^n \\ &= Q_N f_0(t) - \int_0^\infty ds \, \frac{1}{2\pi} \frac{Q_N \, \Delta \mathrm{Im} f_0(s)}{s - t} \left(\frac{t+1}{s+1}\right)^{N+1}. \end{aligned} \tag{19}$$

The second term on the right hand side cancels the higher order terms in the Taylor expansion of $Q_N f_0(t)$ so that the result is a polynomial of degree N. In our case, $\Delta \mathrm{Im} f_0 = -2\pi$, and from (17) and (18) we obtain the following equations for Q_N:

$$\int_0^\infty ds \frac{Q_N}{(s+1)^{m+1}} = 0, \qquad m = N+1, \cdots, 2N \ . \tag{20}$$

This set of conditions for Q_N is most transparent after doing the change of variables $s = (1-x)/(1+x)$,

$$\int_{-1}^{1} dx\, (1+x)^{m-1} Q_N \left(\frac{1-x}{1+x}\right) = 0 \ , \tag{21}$$

and shifting $m \to m' = m - N - 1$,

$$\int_{-1}^{1} dx\, (1+x)^{m'} (1+x)^N Q_N \left(\frac{1-x}{1+x}\right) = 0 \ , \qquad m' = 0, 1, \cdots, N-1. \tag{22}$$

This equation contains a product of two polynomials. The first one can be written as a sum over Legendre polynomials

$$(1+x)^{m'} = \sum_{i=0}^{m'} c_i P_i(x) \ , \tag{23}$$

where c_i are calculable coefficients (that we do not need). The second piece

$$(1+x)^N Q_N \left(\frac{1-x}{1+x}\right) \tag{24}$$

is some polynomial of degree N in x. This can be seen by realizing that $Q_N\left((1-x)/(1+x)\right)$ is a polynomial of degree N in $(1-x)/(1+x)$, and so the factor $(1+x)^N$ removes all powers of $(1+x)$ in the denominator of the expansion of Q_N. Legendre polynomials satisfy the orthogonality relation

$$\int_{-1}^{1} dx\, P_a(x) P_b(x) = 0 \quad \text{for } a \neq b \tag{25}$$

which combined with (22) implies that we can identify the second piece (24) as

$$(1+x)^N Q_N \left(\frac{1-x}{1+x}\right) = P_N(x) \ . \tag{26}$$

This identification is unique since (22) with $m' = N-1$ implies (24) must be a polynomial of at least degree N, but we already showed that (24) is a polynomial of degree exactly N. Rewriting (26) in terms of the original variable s we obtain

$$Q_N(s) = (s+1)^N P_N \left(\frac{1-s}{1+s}\right) \ , \tag{27}$$

up to an overall normalization. The normalization does not affect the computation of the two-point function since it is canceled between the numerator and the denominator in (15).

Next, in order to make the Padé approximation in (15) as accurate as possible, Migdal proposed taking $N \to \infty$ and also the low energy limit $q^2 \ll \mu^2$, in which case the Legendre polynomial reduces to a Bessel function. This is most easily seen by rewriting the Legendre polynomial as a special case of a Jacobi polynomial $P_N(x) = P_N^{(0,0)}(x)$ and using the relation [29],

$$\lim_{N\to\infty} N^{-a} P_N^{(a,b)}\left(\cos\frac{z}{N}\right) = \lim_{N\to\infty} N^{-a} P_N^{(a,b)}\left(1 - \frac{z^2}{2N^2}\right) = \left(\frac{z}{2}\right)^{-a} J_a(z). \tag{28}$$

Therefore, in this limit with $(a,b) \to (0,0)$ we have

$$Q_N(t) \to J_0(2N\sqrt{t}) \equiv Q_\infty(t). \tag{29}$$

For completeness we have provided a derivation of (29) in the appendix. Here we see that, if we think of μ as the UV cutoff of the theory, then in the low energy and $N \to \infty$ limit, a new, infrared scale emerges as μ/N. This infrared scale μ/N is not determined by the Padé approximation. Instead, it acts as a sort of boundary condition: we can take both the UV cutoff μ and the degree N to infinity while keeping their ratio μ/N fixed,

$$\mu \to \infty, \quad N \to \infty, \quad \frac{\mu}{N} = \mu_{ir} = \text{fixed} . \tag{30}$$

In Sec. 5 we will give these scales a concrete physical interpretation in the deconstructed picture.

We can also calculate the numerator in the Padé approximation using (19)

$$\begin{aligned} R_\infty(t) &\equiv Q_\infty(t) f_0(t) - \frac{1}{2\pi}\int_0^\infty ds\, \frac{Q_\infty(s)\,\Delta \mathrm{Im} f_0(s)}{s-t} \\ &= Q_\infty(t) f_0(t) - \frac{1}{2\pi}\int_0^\infty ds\, \frac{(-2\pi) I_0(2N\sqrt{-s})}{s-t} \\ &= J_0(2N\sqrt{t})(\log t) - \pi\, Y_0(2N\sqrt{t}). \end{aligned} \tag{31}$$

Up to an overall constant which has been factored out of (13), the Padé approximation for the polarization $\Sigma(q^2)$ becomes,

$$\Sigma(q^2) \propto q^2 \frac{J_0(2Nq/\mu)\log(q^2/\mu^2) - \pi\, Y_0(2Nq/\mu)}{J_0(2Nq/\mu)}. \tag{32}$$

Here we see that if we identify

$$\frac{1}{\mu} = \epsilon, \quad \frac{2}{\mu_{ir}} = z_0, \tag{33}$$

we recover exactly (9). What is interesting here is that we have obtained the same results as AdS/CFT for the two-point function, without reference to the anti-de Sitter space and holography. Furthermore, in the bulk-to-boundary propagator the correct dependence on the infrared boundary is reproduced in the Padé approximation in the limit $N \to \infty$ and $q^2/\mu^2 \ll 1$. We should emphasize that the $N \to \infty$ limit is different from the large N_c limit in a gauge theory; in fact, Migdal's starting point is already QCD in the large N_c limit.

It is instructive to look at the bulk-to-boundary propagator (6) in AdS/CFT more closely, paying attention to the analytic structure of the Bessel functions. The Bessel function of the first

kind, $J_\nu(x)$, is analytic and has no poles in the complex plane. Near the origin it behaves like $J_\nu(x) \sim x^\nu$. On the other hand, the Bessel function of the second kind $Y_\nu(x)$ has the following series expansion, when ν is an integer,

$$Y_\nu(x) \sim \sum_{m=0}^{\nu-1} a_m \frac{x^{2m}}{x^\nu} + \frac{2}{\pi} J_\nu(x) \log\frac{x}{2} + \sum_{m=0}^{\infty} b_m x^{2m+\nu} \quad . \tag{34}$$

That is $Y_\nu(x)$ has a pole of order ν at the origin and a non-analytic piece $J_\nu(x)\log x$. Thus we see in (6) the terms that are non-analytic in momentum q conspire to cancel each other. Both the numerator and denominator of the two-point function determined by AdS/CFT as in (9) are analytic in the momentum q. AdS/CFT is secretly constructing the Padé approximation to the two-point function. The reason for this goes back to the large N_c limit for which quantum corrections can be ignored in AdS/CFT. There is no source of non-analyticity at large N_c, and resonances become simple poles in the two-point function. What we have found is that AdS/CFT gives the best approximation to the conformal behavior in the UV by means of a set of infinitely narrow resonances.

4 Power Corrections to the Conformal Correlator

In this section we show that, after breaking the conformal symmetry by adding power corrections to the two-point function in (13), Padé approximation continues to reproduce the results of AdS/CFT. The motivation for such considerations stems from the well-known fact that in the deep Euclidean region one can perform OPE to the two-point function of vector currents in QCD, which exhibits a conformal behavior at leading order in q^2. Moreover, at subleading order the OPE contains power corrections arising from various quark and gluon condensates [25, 26]. In the original publications [22, 23] Migdal only considered breaking of the conformal symmetry due to the running of $\alpha_s(\mu)$, the QCD coupling constant. We will see how to incorporate these power corrections in both the AdS/CFT and the Padé approximation. In the context of AdS/QCD, such power corrections were recently considered in [15] from a geometric perspective, which we will follow*.

Power corrections arise from condensates of operators, so AdS/CFT instructs us to turn on a normalizable background for the 5D fields which act as the source for those operators [30, 31]. In order to reproduce the power corrections to correlators in the UV, we can add appropriate higher dimension operators to the 5D Lagrangian, and then determine the consequences of these corrections in the IR [9, 10]. At lowest order in the chiral condensates, in [15] it is shown that power corrections can be included in AdS/CFT instead by power law deviations from the AdS metric near the UV boundary. More specifically, we consider the metric,

$$\begin{aligned} ds^2 &= w(z)^2 \left(\eta_{\mu\nu} dx^\mu dx^\nu - dz^2\right) && (35) \\ w(z) &= \frac{1}{z}\left(1 + O_4'\, z^4 + O_6'\, z^6 + \cdots\right), && (36) \end{aligned}$$

*We are grateful to Veronica Sanz for kindly clarifying the computation in [15].

where O'_{2n} is a quantity with mass dimension $2n$ and is assumed to be small compared to the AdS length scale so that one can perform a perturbative expansion in O'_{2n}. The bulk-to-boundary propagator $V(q,z)$ satisfies the equation,

$$\left(-q^2 - \partial_z^2 - \frac{w'(z)}{w(z)}\partial_z\right) V(q,z) = 0 \tag{37}$$

with the IR boundary condition chosen to be $\partial_z V(q, z_0) = 0$ as before, while the UV boundary condition fixes $V(q,\epsilon)$ to some nonvanishing value. (Recall that the overall normalization of $V(q,z)$ factors out of $\Sigma(q^2)$.) For $w(z) = w_0(z) = 1/z$, the solution $V_0(q,z)$ is given by (6). We are interested in the following metric

$$\begin{aligned} w(z) &= \frac{1}{z}\left(1 + O'_{2n}\, z^{2n}\right) \\ &= w_0(z)\left(1 + w_{2n}(z)\right). \end{aligned} \tag{38}$$

Writing $V(q,z) = V_0(q,z) + V_{2n}(q,z)$, a perturbative expansion in O'_{2n} leads to

$$\left(-q^2 - \partial_z^2 - \frac{w_0'(z)}{w_0(z)}\partial_z\right) V_{2n} = -w_{2n}'(z)\partial_z V_0. \tag{39}$$

Making the change of variable $z \to x = qz$, $V_0(q,z) = V_0(x)$, and $V_{2n}(q,z) \to \tilde{V}_{2n}(x) = (q^{2n}/O'_{2n})V_{2n}(q,z)$, we have

$$\left(1 + \partial_x^2 - \frac{1}{x}\partial_x\right)\tilde{V}_{2n} = 2n\, x^{2n-1}\, \partial_x V_0. \tag{40}$$

A particular solution to the above can be obtained by first solving for the Green's function $G(x,x')$

$$\left(1 + \partial_x^2 - \frac{1}{x}\partial_x\right) G(x,x') = x\,\delta(x-x'), \tag{41}$$

subject to the boundary conditions: $G(q\epsilon, x') = \partial_{x'} G(x, qz_0) = 0$. The UV boundary condition is chosen so that $V(q,\epsilon)$ is unchanged by the perturbation. We can solve $G(x,x')$ in the regions $x < x'$ and $x > x'$ separately, and then match over the delta function, which results in [32]

$$G(x,x') = \frac{\pi}{2}\frac{xx'}{AD - BC}\left[A\, J_1(x_<) + B\, Y_1(x_<)\right]\left[C\, J_1(x_>) + D\, Y_1(x_>)\right] \tag{42}$$

with $x_{<,>} = \{\min, \max\}(x,x')$ and

$$A = Y_1(q\epsilon), \quad B = -J_1(q\epsilon), \quad C = Y_0(qz_0), \quad D = -J_0(qz_0). \tag{43}$$

A particular solution to (40) is then

$$\begin{aligned} \tilde{V}_{2n}(x) &= \int_0^{x_0} dx'\ 2n(x')^{2n-2}(\partial_{x'}V_0)\ G(qz,x') \\ &= \frac{\pi}{2}\frac{V_0(x)}{AD-BC}\int_0^{x} dx'\ 2n(x')^{2n-2}(\partial_{x'}V_0)\ x'[AJ_1(x') + BY_1(x')] \\ &\quad + \frac{\pi}{2}\frac{x[AJ_1(x) + BY_1(x)]}{AD-BC}\int_x^{x_0} dx'\ 2n(x')^{2n-2}(\partial_{x'}V_0)\ V_0(x'), \end{aligned} \tag{44}$$

where $x_0 \equiv qz_0$. Recall that in computing the two-point function (9) we are only interested in the behavior of $V(q,z)$ near the UV boundary, so we focus on the limit $q\epsilon \to 0$ and $x \ll 1$. In this limit,

$$AD - BC \to -\frac{2}{\pi}\frac{1}{q\epsilon}D = \frac{2}{\pi}\frac{J_0(qz_0)}{q\epsilon}, \quad x[AJ_1(x) + BY_1(x)] \to -\frac{2}{\pi}\frac{1}{q\epsilon}xJ_1(x) \approx -\frac{x^2}{\pi}\frac{1}{q\epsilon}. \tag{45}$$

Moreover, if we further consider the energy regime where $x_0 \gg 1$, the leading contribution in (44) is

$$\begin{aligned} \tilde{V}_{2n}(x) &\approx -\frac{\pi}{4}\frac{x^2}{J_0(qz_0)}\int_0^\infty dx'\; 2n(x')^{2n-2}(\partial_{x'}V_0)V_0(x') \\ &= -\frac{\alpha}{2\pi}\; x^2 J_0(qz_0), \end{aligned} \tag{46}$$

where

$$\begin{aligned} \alpha &= \frac{\pi^2}{2J_0(qz_0)^2}\int_0^\infty dx'\; 2n(x')^{2n-2}(\partial_{x'}V_0)V_0(x') \\ &= -\frac{\pi^2}{2J_0(qz_0)^2}\,\frac{1}{2}\int_0^\infty dx'\; 2n(2n-2)(x')^{2n-3}V_0(x')^2 \\ &= (-1)^n\;\frac{\sqrt{\pi}\; n^2\Gamma(n)^3}{\Gamma\left(\frac{1}{2}+n\right)}. \end{aligned} \tag{47}$$

Apart from the factor of $(-1)^n$, α results in the same numerical coefficient as in [15], which, instead of solving for the propagator (42), used the ansatz $\tilde{V}_{2n} = C(x)V_0(x)$ in (40) and then solved for $C(x)$ [33]. The extra factor $(-1)^n$ arises from the fact that we are considering Minkowski momentum, whereas [15] considered the Euclidean momentum. It is worth noting that $\tilde{V}_{2n}$, and hence $V_{2n}(q,\epsilon)$, has the same infrared coefficient, $J_0(qz_0)$, as $V_0(q,\epsilon)$, which implies that the position of the poles in the two-point function $\Sigma(q^2)$ is not altered by the power corrections. That is the mass of the resonances in the CFT is not sensitive to the power corrections at this order in perturbation theory and in the limit $x_0 = qz_0 \gg 1$. Factoring out a $1/g_5^2$, the two-point function now becomes

$$\begin{aligned} \Sigma(q^2) &\approx w_0(1+w_{2n})\left\{\frac{\partial_z V_0(q,z)}{V_0(q,z)} + \frac{\partial_z V_{2n}(q,z)}{V_0(q,z)} - \frac{\partial_z V_0(q,z)}{V_0(q,z)}\,\frac{V_{2n}(q,z)}{V_0(q,z)}\right\}\Bigg|_{z=\epsilon} \\ &\to \frac{1}{\epsilon}\left\{\frac{\partial_z V_0(q,\epsilon) + \partial_z V_{2n}(q,\epsilon)}{V_0(q,\epsilon)}\right\} \\ &= \frac{q^2}{2}\frac{\log(q^2\epsilon^2)J_0(qz_0) - \pi Y_0(qz_0) - (1/q^{2n})\alpha\, O'_{2n}\, J_0(qz_0)}{J_0(qz_0)} \end{aligned} \tag{48}$$

where the power correction $1/q^{2n}$ comes from only the $w_0(\partial_z V_{2n})/V_0$ term. All the other terms involving w_{2n} or V_{2n} vanish in the limit $q\epsilon \to 0$.

Next we discuss how to incorporate the effect of power corrections in Migdal's regularization, which was not discussed in [22, 23]. As a matter of fact, it requires almost no effort to construct the Padé approximation when including the power corrections to the conformal correlator (13),

since the spirit of Padé approximation is to use a ratio of polynomials and the power corrections $1/q^{2n}$ are themselves in the canonical form of Padé approximation. Let us consider the following two-point function

$$\begin{aligned}\Sigma(q^2) &= q^2\left(\log\frac{q^2}{\mu^2}+\frac{O_{2n}}{q^{2n}}\right)\\ &= \Sigma_0(q^2)+\Sigma_{2n}(q^2).\end{aligned} \tag{49}$$

Since we already know that the Padé approximant to Σ_0 is q^2R_∞/Q_∞, as discussed in the previous section, and the correction Σ_{2n} itself is already in the form of a Padé approximant, we conclude that the Padé approximant $q^2R(q^2)/Q(q^2)$ to $\Sigma(q^2)$ is

$$Q(q^2)=q^{2n}Q_\infty(q^2),\quad R(q^2)=q^{2n}R_\infty(q^2)+O_{2n}\,Q_\infty(q^2). \tag{50}$$

Given the expression for Q_∞ and R_∞ in (29) and (31) and the identifications in (33), we obtain for $\Sigma(q^2)$

$$\Sigma(q^2)=q^2\frac{q^{2n}J_0(qz_0)\log(q^2\epsilon^2)-\pi\, q^{2n}\,Y_0(qz_0)+O_{2n}J_0(qz_0)}{q^{2n}J_0(qz_0)}. \tag{51}$$

Therefore we see that the Padé approximation reproduces the result from AdS/CFT in (48) in the limit $qz_0 \gg 1$, identifying $O_{2n}=-\alpha O'_{2n}$. In our approach, the lowest order in O_{2n} simply yields a pole of order n at $q^2=0$ to the Padé approximant which obviously does not change the high energy behavior. However, the perturbative approach that we used here is not valid for $qz_0\approx 1$, and thus the expression above including the spurious $q^2=0$ poles is not a valid description of the low-lying mass spectrum. Indeed, in [15] the shift in the mass for the low-lying hadrons are computed numerically and found to be non-zero.[†]

5 Physical Interpretation

We have shown that Migdal's approach using the Padé approximation reproduces the two-point function obtained from Maldacena's AdS/CFT correspondence. In both approaches the goal was to calculate at strong coupling; what is intriguing is that they both lead to the same approximation of the vector current-current correlator. In this section, we propose an explicit correspondence between these two approaches.

We propose that finite N in the Padé approximation is equivalent to N hidden local symmetries [34, 35] that mock up the two-point function resonances, which in turn is just an N-site deconstruction [27, 28, 36, 38, 39] of the AdS space. In the context of QCD, the idea of considering a large number of hidden local symmetries was discussed in [16] and found to qualitatively and in some cases quantitatively agree with low energy data. Here our observation is that the poles in the Padé approximation can be interpreted as vector resonances. A Padé approximant at finite N order has N poles, which correspond to N vector resonances. Using hidden local

[†]We thank Johanness Hirn and Veronica Sanz for pointing out an erroneous statement about the decay constants in an early version of this paper.

symmetry, each massive vector is reinterpreted as a massive gauge boson of a broken gauge group. Hence, N massive vector resonances can be represented as a product gauge theory with N broken gauge groups. Furthermore, the residues at the poles of the resonances are positive [22], and can be understood as the decay constants of the resonances.

According to our proposal, the resonance masses as determined by the Padé approximation become the Kaluza-Klein masses of a gauge field in an extra dimension, so the product gauge theory described above is equivalent to a deconstructed extra dimension in the limit of a large number of lattice sites. In the deconstructed theory, the inverse lattice spacing roughly corresponds to the the UV scale μ, while the length of the AdS space corresponds to N/μ. The limits (30) are equivalent to holding the length of the AdS space fixed while taking the lattice spacing to zero, which in deconstruction corresponds to obtaining a continuous extra dimension. The masses as determined by the Padé approximation are given by roots of Jacobi polynomials, which may differ from a straightforward latticization of a slice of AdS_5. Hence, the regularization of the 5D gauge theory by a moose model based on the Padé approximation differs from those discussed in [36, 37, 38, 39], although only in the way in which it approaches the continuum theory. Nevertheless, these identifications, as well as the fact that the spectrum obtained from the Padé approximations matches the Kaluza-Klein spectrum of a 5D gauge field in a slice of the AdS_5 geometry, suggest it is natural to identify the large N and low energy limit of the Padé approximation with the continuum limit of a deconstructed radial direction of AdS_5.

In AdS/QCD, a puzzle one might raise is the assumption that QCD, being asymptotically free and renormalizable, is equivalent to a 5D gauge theory, which is neither asymptotically free nor renormalizable. In this regard, Padé approximation in the large N and low energy limit provides a UV completion of the 5D gauge theory, since the linear moose theory is both asymptotically free and renormalizable. In other words, Padé approximation provides a gauge invariant regularization of the 5D theory in which the masses are given by roots of Jacobi polynomials.

We should point out that the argument for the emergence of a deconstructed extra dimension is stronger than a simple identification of vector boson masses. The AdS/CFT prescription for computing the generating functionals in the CFT arises from the deconstructed extra dimension in a natural way, as was first pointed out in [16]. As a result, the deconstructed theory that emerges from the Padé approximation also predicts identical decay constants as the AdS/CFT. Consider the following action for the N-site linear moose model

$$S_{\mathrm{m}} = \int d^4x \sum_{n=1}^{N} v_n^2 \left|D_\mu \Sigma^n\right|^2 - \sum_{n=1}^{N} \frac{1}{4g_n^2} \left(F_{\mu\nu}^n\right)^2 , \tag{52}$$

where the covariant derivative is defined as

$$D_\mu \Sigma^n = \partial_\mu \Sigma^n - i A_\mu^{n-1} \Sigma^n + i \Sigma^n A_\mu^n \tag{53}$$

with $A_\mu^0 = A_\mu^{N+1} = 0$. The current-current correlators can be computed by first gauging the global symmetry on the zero$^{\mathrm{th}}$ site, where the CFT lives, and then differentiating the action

with respect to the gauge field $A^0_\mu \equiv B_\mu$:

$$\langle J_\mu(x) J_\nu(y)\rangle \equiv i\langle 0|T(J_\mu(x)J_\nu(y))|0\rangle = -\frac{\delta^2}{\delta B_\mu(x)\delta B_\nu(y)} \int \mathcal{D}A \; e^{iS_{\rm m}[A,B]} \tag{54}$$

where the gauge field at the boundary $A^0_\mu = B_\mu$ is non-dynamical. In the continuum limit this is equivalent to having a 5D gauge field $A_\mu(x,z)$ with a fixed boundary condition

$$A_\mu(x,0) = B_\mu(x). \tag{55}$$

Furthermore, the action of the N-site moose $S_{\rm m}$ becomes that of a 5D gauge theory in a slice of AdS$_5$. At tree-level, the generating functional in (54) is simply

$$\int \mathcal{D}A \; e^{iS_{\rm m}[A]} \approx e^{iS_{\rm cl}[A^{\rm cl}_\mu]} \tag{56}$$

where $A^{\rm cl}_\mu$ is the solution to the classical field equation with the specified boundary condition (55). This is the same as the AdS/CFT prescription in (2): the generating functional for an operator in the CFT is the same as the classical action of a bulk field whose boundary value serves as the source for the operator.

The use of hidden local symmetry also makes it clear why a global symmetry carried by the vector resonances on the CFT side must become a gauge symmetry in the AdS picture. It is because each of these vector resonances realizes one copy of the hidden local symmetry at each lattice site. Therefore in the continuum limit the gauge symmetry at each site becomes the gauge symmetry in the continuous extra dimension, which is the bulk of AdS. It is worth emphasizing that the equivalence of these two viewpoints is not a consequence of conformal symmetry, as exemplified by the discussion in the previous section when we break the conformal symmetry by adding power corrections.

6 Conclusions and Discussion

Following suggestions by Shifman [20] and Voloshin [21], we have studied the striking similarity between two approaches to estimating the bound state spectrum and decay constants in asymptotically conformal field theories: one that was invented by Migdal in the '70s and the other making use of the AdS/CFT correspondence. The latter approach was recently applied to QCD, the input being the AdS/CFT correspondence and chiral symmetry breaking. The similarity of results based on holography and results based on other approximations may provide additional insight into the predictive success of the AdS/QCD models.

In Migdal's Padé approximation to the conformal two-point correlation functions, the $N^{\rm th}$ order approximation gives rise to a series of N poles whose locations are interpreted as resonance masses; the residues of those poles are related to the decay constants of those resonances. Migdal found that the approach to the logarithmic branch cut as N is increased is through poles whose locations are proportional to zeroes of Jacobi polynomials, which approach Bessel functions in

the large N limit. Power corrections to the correlation function at leading order do not modify the spectrum or the decay constants in the limit $qz_0 \gg 1$. In the AdS/CFT approach, the geometry is approximated by a slice of 5D anti-de Sitter space, whose isometry near the AdS boundary reproduces the conformal symmetry of the corresponding field theory in the UV. Solutions to the equations of motion for 5D fields correspond to resonances. The AdS geometry leads to vector meson masses that are zeroes of Bessel functions, which match the result of the Padé approximation to the conformal two-point functions after identifying the IR scales in both approaches. Power corrections are included by modification of the AdS geometry, which leads to a modification of the decay constants once again in agreement with the Padé approximation.

Migdal's approach is similar in spirit to QCD sum rules [25, 26]: the input is high energy information about QCD in the form of the operator product expansion, together with some assumptions about the dynamics of chiral symmetry breaking and confinement; and the output is low energy information about hadrons such as their spectrum, decay constants and couplings. AdS/QCD is similar in spirit as well, where the prescription for extrapolating between UV and IR is dictated by the dynamics of a higher dimensional field theory in AdS space. We find it remarkable that Migdal's method via the Padé approximation is exactly equivalent to the holographic prescription from AdS/CFT, at least for the vector current two-point function.

We mainly concentrated on two-point functions of vector currents in this paper. It is natural to ask whether the equivalence between the approaches of Migdal and AdS/CFT persists beyond two-point functions[‡] and if from Migdal's approach one could derive the higher dimensional theory in a systematic fashion. In terms of extending the range of validity of a holographic dual of QCD in a phenomenological approach such as AdS/QCD, there are still many hurdles to overcome. For example, a simple argument (see for example [20]) for the Regge trajectory of QCD suggests that at high energies the mass of the N^{th} excited hadron is proportional to $\sqrt{N}$, whereas in AdS/QCD models the mass is generically proportional to N for highly excited states. On the other hand, the bulk metric must always approach AdS near the boundary in order to reproduce the conformal symmetry. It would be interesting to find an AdS/QCD model that would give rise to the required Regge trajectory and the asymptotic conformal behavior at the same time.

In Sec. 5 we saw that, by combining the ideas of hidden local symmetry and deconstruction, the resonances determined by the Padé approximation can be interpreted as arising from a linear moose model. We discussed that the moose model becomes a deconstructed warped extra dimension when the number of resonances becomes large. We further showed that this extra dimension can be identified with the radial direction of anti-de Sitter space. It is interesting to ask what would happen if we applied the same setup to the two-point function of stress-energy tensors in the CFT. In this case, there is a tower of massive spin-2 resonances in the Padé approximation. A massive spin-2 particle can be thought of as a massive graviton, realizing a copy of broken 4D diffeomorphism invariance. Therefore a theory for N massive spin-2 resonances might be interpreted in terms of an N-site latticized extra dimension with gravity. However, a deconstructed gravitational theory is very different from a deconstructed gauge theory as the latticized gravity suffers from a strong coupling problem [41] in such way that the effective N-site

[‡]See [40], for example, for an attempt to generalize Migdal's approach to three-point amplitudes.

theory breaks down at a much lower scale than might naively be expected. It was recently found that in the AdS space, the strong coupling problem is not as severe as in flat space [42, 43]. It would be interesting to consider strong coupling issues from the perspective of the Padé approximation. An approach like this for the spin-2 resonances might lead to a better understanding of how a gravitational theory can be equivalent to a field theory in fewer dimensions without gravity.

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Appendix

In this appendix we provide a derivation of (29). We make use of the identity [29]

$$P_n^{(a,b)}(x) = \frac{\Gamma(n+a+1)}{\Gamma(n+1)\Gamma(a+1)} \left(\frac{1+x}{2}\right)^n \ {}_2F_1\left(-n, -n-b\,; a+1; \frac{x-1}{x+1}\right), \tag{57}$$

where the hypergeometric function is defined as

$${}_2F_1(a,b;c;x) = \sum_{m=0}^{\infty} \frac{\Gamma(a+m)}{\Gamma(a)} \frac{\Gamma(b+m)}{\Gamma(b)} \frac{\Gamma(c)}{\Gamma(c+m)} \frac{x^m}{m!}. \tag{58}$$

In (27) we need to consider

$$\begin{aligned}(1+s)^N P_N^{(a,b)}\left(\frac{1-s}{1+s}\right) &= \frac{\Gamma(N+a+1)}{\Gamma(N+1)\Gamma(a+1)} \ {}_2F_1(-N, -N-b\,; a+1; -s) \\ &= \sum_{m=0}^{\infty} \frac{(-s)^m}{\Gamma(a+1+m)m!} \frac{\Gamma(N+a+1)\Gamma(N+b+1)}{\Gamma(N-m+1)\Gamma(N+b+1-m)} \end{aligned} \tag{59}$$

where we have used the identity

$$\frac{\Gamma(-N+m)}{\Gamma(-N)} = (-1)^m \frac{\Gamma(N+1)}{\Gamma(N-m+1)}. \tag{60}$$

In the $N \to \infty$ limit we can make use of the Sterling's formula

$$\lim_{x\to\infty} \Gamma(x) = e^{-x} x^{x-1/2} \sqrt{2\pi}. \tag{61}$$

Then it is straightforward to show that in (59)

$$\lim_{N\to\infty} \frac{\Gamma(N+a+1)\Gamma(N+b+1)}{\Gamma(N-m+1)\Gamma(N+b+1-m)} \longrightarrow N^{2m+a} \quad , \tag{62}$$

which leads to

$$\begin{aligned} Q_a(t) &= N^a \sum_{m=0}^{\infty} \frac{(-4N^2 t)^m}{2^{2m}\Gamma(a+m+1)m!} \\ &= t^{-a/2} J_a(2N\sqrt{t}). \end{aligned} \tag{63}$$

For $a = 0$ we obtain (29).

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